UNIVERSITY OF LONDON

BA EXAMINATION

for Internal Students

This paper is also taken by Combined Studies Students

PHILOSOPHY

Optional Subject (j): Mathematical Logic

Answer THREE questions.

1. i. For Σ a set of sentences of Sentential Logic it holds that "if $\Sigma \models \phi$, then there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \phi$". Use this to prove the compactness theorem.

   ii. Assume that every finite subset of Σ is satisfiable. Show that, for any propositional sentence α, the same holds for at least one of $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$.

   iii. Let Δ be a set of sentences of Sentential Logic such that every finite subset of Δ is satisfiable and for every sentence ϕ, $\phi \in \Delta$ or $\neg\phi \in \Delta$. Define the truth valuation v by $v(P) = \top$ if $P \in \Delta$ and $v(P) = \bot$ if $P \notin \Delta$ for each propositional variable P. Show that for every sentence ϕ we have: $\overline{v}(\phi) = \top$ iff $\phi \in \Delta$.

2. (a) Show the following facts, using the Truth Definition.

   (i) $\alpha \wedge \exists x\beta$ and $\exists x(\alpha \wedge \beta)$ are logically equivalent if x does not occur free in α.

   (ii) $\exists x(Ax \rightarrow \forall xAx)$ is true in all structures.

   (iii) $\exists x(\exists xAx \rightarrow Ax)$ is true in all structures.

TURN OVER

3. The axiom groups of First Order Predicate Logic with Identity, $FOL =$, are all universal generalisations of formulas of the following form:

   1. All tautologies
   2. $\forall x\alpha \rightarrow \alpha^x_t$ where t is substitutable for x in α
   3. $\forall x(\alpha \rightarrow \beta) \rightarrow (\forall x\alpha \rightarrow \forall x\beta)$
   4. $\alpha \rightarrow \forall x\alpha$ where x does not occur free in α
   5. $x = x$
   6. $x = y \rightarrow (\alpha^z_x \rightarrow \alpha^z_y)$

   Let $\vdash$ be the logical consequence relation of $FOL =$ obtained from these axiom groups together with the inference rule Modus Ponens.

   (a) What does it mean that t is 'substitutable' for x in α?

   (b) Show that if $\vdash \alpha \rightarrow \beta$ then $\vdash \forall x\alpha \rightarrow \forall x\beta$

   (c) Show (by induction on the length of a proof) that if $\Gamma \vdash \alpha$ and x does not occur free in any formula of Γ then $\Gamma \vdash \forall x\alpha$

4. Let s_1 and s_2 be (valuation) functions from the set of variables V of a first order language into $|\mathcal{A}|$. Let s_1 and s_2 agree on all the free variables of the formula ϕ.

   (a) show, by induction on the degree of t, that if t is a term of ϕ then $\overline{s_1}(t) = \overline{s_2}(t)$.

   (b) show, by induction on the degree of ϕ, that $\models_{\mathcal{A}} \phi[s_1]$ iff $\models_{\mathcal{A}} \phi[s_2]$.

5. (a) State the completeness theorem for first order predicate logic with identity.

   (b) State the compactness theorem for first order predicate logic with identity and derive it from the completeness and soundness theorems.

   (c) State the Löwenheim-Skolem theorem for first order predicate logic and derive it from the completeness theorem.

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(d) Show that if Γ is a set of sentences of any language and Γ has only infinite models, then Γ has infinite models of any cardinality [you may assume that a consistent set of sentences remains consistent if its language is extended by the addition of any cardinality of new constants].

6. Let $\mathfrak{N}$ be the standard structure of natural numbers with domain N. For every $n \in N$ let $\mathbf{s}_n$ be the name of n in the first-order language of arithmetic and let y be a fixed variable.

   (i) Show that
   $$\Sigma = Th(\mathfrak{N}) \cup \{\mathbf{s}_n < y \mid n \in N\}$$
   is satisfiable (where $Th(\mathfrak{N})$ is the set of sentences true on the standard structure of natural numbers).

   (ii) Let ω be the rule

   "If $\Gamma \vdash \phi^x_{s_n}$ for every $n \in N$, then $\Gamma \vdash \forall x\phi$."

   Show that Σ is inconsistent when this rule is added to the Predicate Calculus.

7. Let Γ be a set of sentences in the language of arithmetic that is maximal w.r.t. the property that $\Gamma \nvdash \alpha$ for some fixed α (maximality means here that if $\beta \notin \Gamma$ then $\Gamma \cup \{\beta\} \vdash \alpha$).

   (i) Show that Γ is complete.

   (ii) Given that $A_E \subseteq \Gamma \subseteq Th(\mathfrak{N})$, outline a proof that Γ is not axiomatizable (where A_E is a finite axiomatization of a subtheory of $Th(\mathfrak{N})$ in the full language of arithmetic and $Th(\mathfrak{N})$ is the set of sentences true on the standard structure of natural numbers).

8. Outline a proof of Tarski's Undefinability Theorem and discuss its connection with Gödels Incompleteness Theorem.

9. i. Let Γ be a recursive set of sentences. Argue that the relation $\mathrm{Ded}_\Gamma(x, y)$ given by

   $Ded_\Gamma(x, y) \iff x$ is a the code of a sentence and y is the code of sequence-of-formulas that constitutes a deduction of that sentence from Γ, is recursive.

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ii. Outline a proof that a relation P is recursively enumerable if and only if it is weakly representable in A_E.

10. Given that A_E is a finitely axiomatized theory in the first-order language of arithmetic, such that every recursive relation is representable in A_E, prove:

    i. For any recursive relation R there is a formula that represents R in any theory Σ in the language of arithmetic such that $A_E \cup \Sigma$ is consistent.

    ii. If Σ is a theory in the language of arithmetic such that $A_E \cup \Sigma$ is consistent, then Σ is not recursively decidable.

END OF PAPER