UNIVERSITY OF LONDON

BA EXAMINATION

for Internal Students

This paper is also taken by Combined Studies Students

PHILOSOPHY

Optional Subject (i): Set Theory and Further Logic

Answer THREE questions, at least ONE from EACH section

SECTION A

1. Let an 'object' be any set or atom (ur-element).

   (i) Refute the claim that for any unary (1-place) predicate in the language of set theory there is a set whose members are exactly the objects which satisfy that predicate.

   (ii) State the Axiom Schema of Separation and use it to show that there is no set which has every object as a member.

   (iii) Assuming that there is a set, show that not every set has a member.

   (iv) Show that no two sets have exactly the same members.

   (v) Show that for any object there is a set whose sole member is that object.

2. (i) Define the Cartesian Product of A with B, $A \times B$. Under what conditions is a subset of $A \times B$ (a) a function, (b) an injection, (c) a surjection on B (d) a bijection from A to B ?

   (ii) Define the identity function on a set C, id_C. Show that it is a bijection from C to C.

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(iii) Let f be an injection with domain A and range B. Define the inverse of f, f^{-1}, and show that it is a bijection from B to A.

(iv) Let f and g be functions such that $dom(f) \subseteq ran(g)$. Define the composition of f with g, $f \circ g$. Show that if f is a bijection from B to C and g is a bijection from A to B, $f \circ g$ is a bijection from A to C

(v) Let K be a set of sets. Let $\sim_K$ be the relation on K (i.e. subset of $K \times K$) such that for any $y, z \in K, y \sim_K z$ if and only if there is a bijection from y to z. Show that $\sim_K$ is an equivalence relation.

3. For any S, let '$|S|$' denote the cardinality of S. Assume that $|A| = |B|$ if and only if there is a bijection from A to B, and that $|A| \leq |B|$ if and only if there is an injection from A to B.

   (i) Show that if $|A| = |D|$ and $|B| = |E|$, $|A \times B| = |D \times E|$; show that if also $A \cap B = \emptyset = D \cap E$, $|A \cup B| = |D \cup E|$

   (ii) Define cardinal addition $(+)$ and cardinal multiplication $(\cdot)$. Show that both are commutative: for any sets $A, B, |A|+|B| = |B|+|A|$ and $|A| \cdot |B| = |B| \cdot |A|$.

   (iii) Show that if $|A| \leq |B|$, $|C| + |A| \leq |C| + |B|$.

   (iv) Let $A \cap B = \emptyset$. Show that $(|A| + |B|) \cdot |C| = (|A| \cdot |C|) + (|B| \cdot |C|)$.

4. (i) What is it for a binary relation R to be (a) a partial ordering, (b) a total ordering, (c) a well-ordering of a set A? Give an example of a set A and a binary relation R on A that totally orders A but does not well-order it.

   (ii) What is it for set to be a *transitive* set (not: transitive relation)? What is it for a set to be an *ordinal*? Given that any member of an ordinal is an ordinal and that any transitive set of ordinals is an ordinal, prove that there is no set of all ordinals.

   (iii) Prove the least ordinal principle: For any unary (1-place) predicate in the language of set theory, if there is an ordinal that satisfies that predicate, there is an $\in$-least ordinal that satisfies it.

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5. (i) What is the von Neumann successor Z^+ of a set Z? What is an inductive set? What is a natural number? Prove that there is exactly one set whose members are just the natural numbers.

   (ii) Writing "ω" for the set of natural numbers, prove that ω is an inductive set.

   (iii) Prove that for any set B if B is inductive $\omega \subseteq B$.

   (iv) State the principle of proof by induction on the natural numbers. Use this principle to show that for any $n \in \omega,\ n \subseteq \omega$.

6. (i) Assume that for any set A there is an infinite cardinal not in A. Define (by transfinite recursion on the ordinals) the aleph operator $\aleph\alpha$. Show that for any ordinal α, $\aleph\alpha \notin \{\aleph\gamma : \gamma < \alpha\}$

   (ii) Prove that for any ordinals α and β, if $\alpha < \beta$ then $\aleph\alpha < \aleph\beta$.

   (iii) Use the fact to be proved in (ii) to show that the predicate "$x = \aleph y$ & y is an ordinal" is univalent, i.e. that for any x, y_1, y_2, if ($x = \aleph y_1$ & y_1 is an ordinal) and ($x = \aleph y_2$ & y_2 is an ordinal), then $y_1 = y_2$.

   (iv) State the Axiom Schema of Replacement. Assuming that there is no set of all ordinals, use Replacement and the fact to be proved in (iii) to show that there is no set of all alephs, i.e. no set $\{\aleph y :$ y is an ordinal$\}$.

7. (i) Show that for any set A, $|A| < |PA|$, where $|A|$ is the cardinal number of A and PA is the power set of A.

   (ii) If $B \subseteq A$, what is the *characteristic function* of B in A? Define cardinal exponentiation κ^μ for cardinals κ, μ. Recalling that $2 = \{0, 1\}$, show that for any set $A, |PA| = 2^{|A|}$.

   (iii) Let B be the set of ω-sequences of 0s and 1s, i.e., the set of functions g with domain ω, such that for any $n \in \omega, g(n) \in 2$. Show that $|\omega| < |B|$.

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(iv) Hilbert's Hotel in Göttingen has $\aleph_0$ rooms and on the day before Chrsimas Eve they are all occupied by people who have booked for at least a week. Then $\aleph_0$ more people show up on Chrismas Eve, each wanting a room for herself or himself. How does the manager accomodate them all? Why, if $2^{\aleph_0}$ people had turned up each wanting his/her own room, would it have been impossible to accommodate them all?

SECTION B

8. Use the propositional tableau proof systems to prove the following three formulas

   (a) $\Diamond(\Box\phi \supset \Diamond\psi) \supset \Diamond(\phi \supset \psi)$ in the system **S**4,

   (b) $\Box(\Box\phi \supset \Box\psi) \vee \Box(\Box\psi \supset \Box\phi)$ in the system **S**5,

   (c) $\Box(\Diamond\phi \supset \psi) \equiv \Box(\phi \supset \Box\psi)$ in the system **S**5.

   Use the constant domain tableau system to determine whether (d) is valid on all **K** models with constant domain

   (d) $(\Diamond(\forall x)A(x) \wedge \Box(\exists x)B(x)) \supset (\exists x)\Diamond(A(x) \wedge B(x))$

   Use the variable domain tableau system to determine whether (e) is valid on all **K** models with varying domain

   (e) $(\Diamond(\forall x)A(x) \wedge \Box(\exists x)B(x)) \supset \Diamond(\exists x)(A(x) \wedge B(x))$.

9. (i) Is the following valid in T: $\Diamond(p \wedge q) \equiv (\Diamond p \wedge \Diamond q)$? Justify your answer.

   (ii) Show that the characteristic thesis of the Brouwerian system, $p \supset \Box\Diamond P$, is not valid in S4.

   (iii) (a) Provide a model to show that $(\Box p \supset \Box\Diamond\Box p)$ is not valid in T.

   (b) Provide a model that shows that $(\Box\Diamond\Box p \supset \Box p)$ is not valid in S4.

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(iv) Provide an S5 model in which $\Box\exists x Px \supset \exists x\Box Px$ is false at some world.

(v) Explain why, if domains are allowed to vary freely, the converse of the Barcan Formula is not valid even in T.

10. (i) Let '$\mathbf{C}P$' abbreviate 'It is contingently true that P'. Define $\mathbf{C}P$ in terms of one of the standard modal operators and explain why $\mathbf{C}P \supset \Box\mathbf{C}P$ is unacceptable.

(ii) What are the modal systems S4 and S5? Under what conditions on the accessibility relation on a model M for propositional modal logic is M a model of (a) S4, (b) S5?
Show that $\Diamond\Box P \supset \Box P$ is true at every possible world of any S5 model but false at some possible world of some S4 model.

11. Let three frames $(G_1, R_1), (G_2, R_2).(G_3, R_3)$, be defined by

$G_1 = \{a, b, c\},\ R_1 = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}$
$G_2 = \{a, b, c\},\ R_2 = \{\langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle c, b\rangle, \langle a, c\rangle, \langle c, a\rangle\}$
$G_3 = \{a, b, c\},\ R_3 = \{\langle a, b\rangle, \langle c, b\rangle, \langle a, c\rangle, \langle c, a\rangle, \langle a, a\rangle\}$

($\langle x, y\rangle$ in R_i means xR_iy). For each of the three fames determine which of the following formulas is/are valid on the frame.

(a) $\Box P \supset P$

(b) $\Box P \supset \Box\Box P$

(c) $P \supset \Box\Diamond P$

(d) $\Box P \supset \Diamond P$

Moreover, if one of the formulas ϕ is not valid on frame (G_i, R_i), give a world x in G_i and a forcing relation $\vdash$ between G_i and $\{P\}$ such that $x \vdash \neg\phi$.

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12. Let (G, R) be the following frame: $G = \{0, 1, 2, 3 \ldots\}$, the set of natural numbers, and nRm holds if n is a smaller number than m. Let $(G, R, \vdash)$ be a model over (G, R). In such a model, '$n \vdash P$', means that natural number n has some property ϕ_P, that is, the model interprets P as a specific property of natural numbers (eg., "n is smaller than a given number a", "n is a multiple of 3", "n is $4, 7$ or 18", etc.).

    (i) For each of the following formulas give a property Φ_P of natural numbers such that the formula is valid on the model $(G, R, \vdash)$

        (a) $P \supset \Box\Diamond P$
        (b) $P \supset \Box P$
        (c) $\Box(P \supset \Box\neg P)$

    (ii) Argue that $(\Diamond P \wedge \Diamond Q) \supset \Diamond((\Diamond P \wedge Q) \vee (P \wedge \Diamond Q) \vee (P \wedge Q))$ is valid on the frame (G, R)

    (iii) Let nQm mean that number n is larger than number m. Give a formula that is valid on the frame (G, Q) but not on (G, R).

13. a. Show that there is a constant domain model allowing predicate abstraction in which $\Diamond\langle\lambda x.P(x)\rangle(c) \wedge \neg\langle\lambda x.\Diamond P(x)\rangle(c)$ can be satisfied.

    b. Show that $(\forall y)\Diamond\langle\lambda x.P(x)\rangle(y) \wedge \neg\langle\lambda x.\Diamond P(x)\rangle(c)$ cannot be satisfied in such a model.

14. Considering $\Box, \Diamond$ as temporal operators, where $\Box P$ is interpreted as "P is and will always be the case" give a formalization using predicate abstraction of the sentence "Someday the president of the Unites States won't be the president of the United States (anymore)" and discuss how a logical contradiction is avoided.

END OF PAPER