UNIVERSITY OF LONDON 088 0310

BA EXAMINATION

for Internal Students

This paper is also taken by Combined Studies Students

PHILOSOPHY

Optional Subject (j): Mathematical Logic

Thursday, 8 May 2008:14.30-17.30

Answer THREE questions.

1. i. For Σ a set of sentences of Sentential Logic it holds that "if $\Sigma \models \phi$, then there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \phi$". Use this to prove the compactness theorem.

   ii. Assume that every finite subset of Σ is satisfiable. Show that, for any propositional sentence α, the same holds for at least one of $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$.

   iii. Let Δ be a set of sentences of Sentential Logic such that every finite subset of Δ is satisfiable and for every sentence ϕ, $\phi \in \Delta$ or $\neg\phi \in \Delta$. Define the truth valuation v by $v(P) = \top$ if $P \in \Delta$ and $v(P) = \bot$ if $P \notin \Delta$ for each propositional variable P. Show that for every sentence ϕ we have: $\overline{v}(\phi) = \top$ iff $\phi \in \Delta$.

2. i. State the soundness theorem for Predicate Logic and show how this theorem entails that the Predicate Calculus is consistent.

   ii. State the completeness theorem for the Predicate calculus and show how it follows from the Consistency Lemma: "every consistent set is satisfiable."

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iii. Prove the compactness theorem: if Φ is a set of formulas such that every finite subset of Φ is satisfiable, then Φ is satisfiable. (You may assume the soundness and completeness of the Predicate Calculus).

3. Show the following facts, using the Truth Definition.

   (i) $\alpha \rightarrow \forall x\beta$ and $\forall x(\alpha \rightarrow \beta)$ are logically equivalent if x does not occur free in α.

   (ii) $\exists x(Ax \rightarrow \forall xAx)$ is true in all structures.

   (iii) $\exists x(\exists xAx \rightarrow Ax)$ is true in all structures.

4. Suppose $\phi_1, \ldots, \phi_n$ is a deduction of Ax from Φ where A is a unary predicate symbol and the variable x does not occur free in Φ. Prove that $\Phi \vdash \forall xAx$ by induction on n.
   In your proof you may use all tautologies plus the axioms

   (i) $\forall x(\alpha \rightarrow \beta) \rightarrow (\forall x\alpha \rightarrow \forall x\beta)$

   (ii) $\alpha \rightarrow \forall x\alpha$, for x not free in α

   (iii) $\forall x_1 \forall x_2 \ldots \forall x_n \alpha$ for any variables $x_1 \ldots x_n$ and any axiom α of the Predicate Calculus.

5. Let $\mathcal{L}$ be the first order language of arithmetic and let the binary predicate letter '$<$' be defined by $t < s \equiv_{df} \exists y(y \not\approx 0 \wedge t+y \approx s)$. Consider the set

$$\Sigma_1 = \{\forall x \neg(x < x), \forall x\forall y\forall z((x < y \wedge y < z) \rightarrow x < z)\}$$

   (i) Show, by giving a deduction, that $\Sigma_1 \vdash \forall x\forall y(x < y \rightarrow \neg y < x)$ (Hint: show that $\Sigma_1 \cup \{(a < b), (b < a)\}$ is inconsistent for a, b not occurring in Σ_1 and proceed from there.)

   (ii) Let $\Sigma_2 = \Sigma_1 \cup \{\forall x\exists y(x < y)\}$
   Argue that Σ_2 has no finite models.

6. A formula is said to be *prenex* when it is of the form

$$Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \alpha$$

where each Q_i is either $\forall$ or $\exists$ and α contains no quantifiers.

(a) Let L be a first order language with $\rightarrow$ as its only sentential connective. If α is a formula of L containing $n+1$ instances of the quantifier symbols, show (by induction on the degree of α) that there is a logically equivalent formula of the form $Qx\beta$ (where Q is $\forall$ or $\exists$) such that β contains only n occurrences of the quantifier symbols.
You may assume the bound variables may be freely renamed and you may assume the following theorems: $\neg\forall x\phi \leftrightarrow \exists x \neg\phi$; $\neg\exists x\phi \leftrightarrow \forall x\neg\phi$. And provided x does not occur free in ψ, you may assume: $(\psi \rightarrow \forall x\phi) \leftrightarrow \forall x(\psi \rightarrow \phi)$; $(\psi \rightarrow \exists x\phi) \leftrightarrow \exists x(\psi \rightarrow \phi)$; $(\forall x\phi \rightarrow \psi) \leftrightarrow \exists x(\phi \rightarrow \psi)$; $(\exists x\phi \rightarrow \psi) \leftrightarrow \forall x(\phi \rightarrow \psi)$.

(b) Conclude that for every formula of L there is a logically equlivalent prenex formula of L.

7. Let $\mathfrak{N}$ be the standard structure of natural numbers with domain N. For every $n \in N$ let s_n be the name of n in the first-order language of arithmetic and let y be a fixed variable.

(i) Show that

$$\Sigma = Th(\mathfrak{N}) \cup \{s_n \not\approx y \mid n \in N\}$$

is satisfiable (where $Th(\mathfrak{N})$ is the set of sentences true in the standard structure of natural numbers).

(ii) Let ω be the rule

If $\Gamma \vdash \phi^x_{s_n}$ for every $n \in N$, then $\Gamma \vdash \forall x\phi$.

Show that Σ is inconsistent when this rule is added to the Predicate Calculus.

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8. Let Γ be a set of sentences in the language of arithmetic that is maximal w.r.t. the property that $\Gamma \not\vdash \alpha$ for some fixed α (maximality means here that if $\beta \notin \Gamma$ then $\Gamma \cup \{\beta\} \vdash \alpha$).

   (i) Show that Γ is complete.

   (ii) Given that $A_E \subseteq \Gamma \subseteq Th(\mathfrak{N})$, outline a proof that Γ is not axiomatizable (where $Th(\mathfrak{N})$ is the set of sentences true on the standard structure of natural number).

9. Outline a proof of Tarski's Undefinability Theorem and discuss its connection with Gödels Incompleteness Theorem.

10. i. Let Γ be a recursive set of sentences. Argue that the relation $Ded_\Gamma(x, y)$ given by

    $Ded_\Gamma(x, y) \iff x$ is the code of a sentence and y is the code of sequence of formulas that constitutes a deduction of that sentence from Γ,

    is recursive.

    ii. Outline a proof that a relation P is recursively enumerable if and only if it is weakly representable in A_E.

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