UNIVERSITY OF LONDON

BA EXAMINATION

for Internal Students

This paper is also taken by Combined Studies Students

PHILOSOPHY

Optional Subject (h): Mathematical Logic

Answer THREE questions.

1. i. For Σ a set of sentences Sentential Logic it holds that "if $\Sigma \models \phi$, then there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \phi$". Use this to prove the compactness theorem.

   ii. Assume that every finite subset of Σ is satisfiable. Show that, for any propositional sentence α, the same holds for at least one of $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$.

   iii. Let Δ be a set of sentences of Sentential Logic such that every finite subset of Δ is satisfiable and for every sentence ϕ, $\phi \in \Delta$ or $\neg\phi \in \Delta$. Define the truth valuation v by $v(P) = \top$ if $P \in \Delta$ and $v(P) = \bot$ if $P \notin \Delta$ for each propositional variable P. Show that for every sentence ϕ we have: $\overline{v}(\phi) = \top$ iff $\phi \in \Delta$.

2. i. State the soundness theorem for Predicate Logic and show how this theorem entails that the Predicate Calculus is consistent.

   ii. State the completeness theorem for the Predicate calculus and show how it follows from the Consistency Lemma: "every consistent set is satisfiable."

   iii. Prove the compactness theorem: if Φ is a set of formulas such that every finite subset of Φ is satisfiable, then Φ is satisfiable.

   You may assume the soundness and completeness of the Predicate Calculus.

TURN OVER

3. Show the following facts, using the Truth Definition.

   (i) $\alpha \to \exists x\beta$ and $\exists x(\alpha \to \beta)$ are logically equivalent if x does not occur free in α.

   (ii) $\exists x(Ax \to \forall x Ax)$ is true in all structures.

   (iii) $\exists x(\exists x Ax \to Ax)$ is true in all structures.

4. Suppose $\phi_1, \ldots, \phi_n$ is a deduction of $B \to Ax$ from Φ where A is a unary predicate symbol and the variable x does not occur free in Φ or B. Prove that $\Phi \vdash B \to \forall x Ax$ by induction on n (i.e., show that $\Phi \vdash B \Rightarrow \forall x\phi_k$ for all $k, 1 \leq k \leq n$.).
   In your proof you may use all tautologies plus the axioms

   (i) $\forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)$

   (ii) $\alpha \to \forall x\alpha$, for x not free in α

   (iii) $\forall x_1 \forall x_2 \ldots \forall x_n \alpha$ for any variables $x_1 \ldots x_n$ and α any axiom of the Predicate Calculus.

5. Let $\mathcal{L}$ be the first order language of arithmetic and let the binary predicate letter '$<$' be defined by $t < s \equiv_{df} \exists y(y \not\approx 0 \ \wedge \ t + y \approx s)$. Consider the set

$$\Sigma_1 = \{\forall x \neg(x < x), \forall x \forall y \forall z((x < y \wedge y < z) \to x < z)\}$$

   i. Show, by giving a deduction, that $\Sigma_1 \vdash \forall x \forall y(x < y \to \neg y < x)$
   (Hint: show that $\Sigma_1 \cup \{(a < b), (b < a)\}$ is inconsistent for a, b not occurring in Σ and proceed from there.)

   ii. Let $\Sigma_2 = \Sigma_1 \cup \{\forall x \exists y(x < y)\}$
   Argue that Σ_2 has no finite models.

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6. Let $\mathfrak{N}$ be the standard structure of natural numbers with domain N. For every $n \in N$ let $\mathbf{s}_n$ be the name of n in the first-order language of arithmetic and let y be a fixed variable.

   (i) Show that
   $$\Sigma = Th(\mathfrak{N}) \cup \{\mathbf{s}_n < y \mid n \in N\}$$
   is satisfiable (where $Th(\mathfrak{N})$ is the set of sentences true on the standard structure of natural number).

   (ii) Let ω be the rule

   "If $\Gamma \vdash \phi^x_{s_n}$ for every $n \in N$, then $\Gamma \vdash \forall x \phi$."

   Show that Σ is inconsistent when this rule is added to the Predicate Calculus.

7. Let Γ be a set of sentences in the language of arithmetic that is maximal w.r.t. the property that $\Gamma \not\vdash \alpha$ for some fixed α (maximality means here that if $\beta \notin \Gamma$ then $\Gamma \cup \{\beta\} \vdash \alpha$).

   (i) Show that Γ is complete.

   (ii) Given that $A_E \subseteq \Gamma \subseteq Th(\mathfrak{N})$, outline a proof that Γ is not axiomatizable (where $Th(\mathfrak{N})$ is the set of sentences true on the standard structure of natural number).

8. Outline a proof of Tarski's Undefinability Theorem and discuss its connection with Gödels Incompleteness Theorem.

9. i. Let Γ be a recursive set of sentences. Argue that the relation $\mathrm{Ded}_\Gamma(x, y)$ given by

   $Ded_\Gamma(x, y) \iff x$ is a the code of a sentence and y is the code of sequence-of-formulas that constitutes a deduction of that sentence from Γ,

   is recursive.

   ii. Outline a proof that a relation P is recursively enumerable if and only if it is weakly representable in A_E.

TURN OVER

10. i. Show how to construct for every positive integer n a first-order sentence α_n containing no extralogical symbols, such that, for every first-order structure $\mathcal{U}$, $\models_{\mathcal{U}} \alpha_n$ iff the domain of $\mathcal{U}$ has exactly n elements.

ii. Let Σ be a set of first-order sentences that has no infinite models. Prove that there exists a positive integer n such that Σ has no models whose domain contains more than n elements. State clearly the main result you use in your proof.

iii. Find a first-order sentence β, containing a binary predicate symbol P, but no other predicate symbols, no individual constants and no function symbols, such thay β has no finite model and such that for any infinite set U there is a model $\mathcal{U}$ of β having U as domain. Justify your answer.

11. Given that A_E is a finitely axiomatized theory in the first-order language of arithmetic, such that every recursive relation is representable in A_E, prove:

i. For any recursive relation R there is a formula that represents R weakly in any theory Σ in the language of arithmetic such that $A_E \cup \Sigma$ is consistent.

ii. If Σ is a theory in the language of arithmetic such that $A_E \cup \Sigma$ is consistent, then Σ is not recursively decidable.

END OF PAPER