UNIVERSITY OF LONDON

BA EXAMINATION

for Internal Students

This paper is also taken by Combined Studies Students

PHILOSOPHY

Optional Subject (j): Mathematical Logic

Answer THREE questions.

1. Let $\vdash$ be a logical consequence relation for a language containing implication $\rightarrow$ and negation $\neg$ where

   $\vdash \alpha$ if α is a tautology
   $\alpha; \alpha \rightarrow \beta \vdash \beta$ (Modus Ponens)

   (a) Prove (by induction on the length of a proof) that the *deduction theorem* holds for $\vdash$, i.e. prove that if $\Gamma; \alpha \vdash \beta$ then $\Gamma \vdash \alpha \rightarrow \beta$. You may wish to use the fact that the following are tautologies:

   $\alpha \rightarrow \alpha$
   $\alpha \rightarrow \beta \rightarrow \alpha$
   $(\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$

   (b) Let $\vdash'$ be a logical consequence relation that satisfies Modus Ponens, the Deduction Theorem and the Cut Rule:

   If $\Gamma \vdash' \alpha_1 \ldots \Gamma \vdash' \alpha_n$ and $\{\alpha_1 \ldots \alpha_n\} \vdash' \beta$, then $\Gamma \vdash' \beta$

   Show that the tautologies mentioned in question 1a are theorems of $\vdash'$

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2. Let Γ_0 be a consistent set of sentences in a language containing only sentential connectives and atomic proposition symbols (i.e. the language of sentential logic). Show that Γ_0 can be extended to a maximal consistent set of sentences Γ'. [Hint: consider an enumeration of all senetences of the language, then construct a set Γ' in stages $\Gamma_0, \Gamma_1 \ldots$ so that Γ' is the union of all the Γ_i. Then verify that Γ' is indeed maximal and consistent.]

3. The axiom groups of First Order Predicate Logic with Identity $FOL =$ are all universal generalisations of (substitution instances) of:

   1. Any tautology
   2. $\forall x\alpha \to \alpha^x_t$ where t is substitutable for x in α
   3. $\forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)$
   4. $\alpha \to \forall x\alpha$ where x does not occur free in α
   5. $x = x$
   6. $x = y \to (\alpha^z_x \to \alpha^z_y)$

   Let $\vdash$ be the logical consequence relation of $FOL =$ obtained from these axiom groups together with the inference rule Modus Ponens.

   (a) What does it mean that t is 'substitutable' for x in α?

   (b) Show that if $\vdash \alpha \to \beta$ then $\vdash \forall x\alpha \to \forall x\beta$

   (c) Show (by induction on the length of a proof) that if $\Gamma \vdash \alpha$ and x does not occur free in any formula of Γ then $\Gamma \vdash \forall x\alpha$

4. Let s_1 and s_2 be (valuation) functions from the set of variable V of a first order language into $|\mathcal{A}|$. Let s_1 and s_2 agree on all the free variables of the formula ϕ.

   (a) show, by induction on the degree of t, that if t is a term of ϕ then $\overline{s_1}(t) = \overline{s_2}(t)$.

   (b) show, by induction on the degree of ϕ, that $\models_{\mathcal{A}} \phi[s_1]$ iff $\models_{\mathcal{A}} \phi[s_2]$.

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5. (a) Show that if $\Gamma \vDash_{\mathcal{A}} \phi$ then $\Gamma \vDash_{\mathcal{A}} \forall x \phi$

   (b) State the soundness theorem for First order Predicate Logic with identity and show that $\vDash_{\mathcal{A}} \phi$ for every ϕ that is a member of an axiom group of $FOL =$ mentioned in question 3. Conclude that $FOL =$ is consistent.

6. Let Γ be a set of sentences of a countable first order langauge L and let $\vdash$ be the familiar logical consequence relation of first order logic. Let L_c be obtained from L by the addition of an ordered countable infinity of new constant symbols $\{c_1 \ldots c_2 \ldots\}$. Suppose also that we have constructed for every formula ϕ of L_c, a formula θ_k of the form

$$\exists x \phi \rightarrow \phi^x_{c_n}$$

   where $n > k$ for every (new) constant c_k occuring in ϕ.

   (a) Show that if $\Gamma \cup \{\theta_1 \ldots \theta_{m+1}\}$ is inconsistent then so is $\Gamma \cup \{\theta_1 \ldots \theta_m\}$. [You may use the facts that if $\Gamma; \alpha \vdash$ (i.e. $\Gamma \cup \{\alpha\}$ is inconsistent) then $\Gamma \vdash \neg\alpha$, and that if $\Gamma \vdash \alpha^x_c$ and c does not occur in Γ or α, then $\Gamma \vdash \forall x \alpha$.]

   (b) Let Θ be the set of all θ_n (for all n). Given that Γ is consistent in the new language L_c conclude that $\Gamma \cup \Theta$ is consistent.
   [Hint: remember that if a set Δ is inconsistent then there is a deduction of a contradiction from *finitely* many premises from Δ.]

7. (a) State the completeness theorem for first order predicate logic with identity.

   (b) State the compactness theorem for first order predicate logic with identity and derive it from the completeness and soundness theorems.

   (c) State the Löwenheim-Skolem theorem for first order predicate logic and derive it from the completeness theorem.

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(d) Let C be a set of κ many constants containing κ. Let L be a language containing the constants of C. Let Γ be a consistent set of sentences of L not containing any member of C. Let Δ be the set

$$\{\neg(c = c') : \ c, c' \text{ are distinct constants in } C\}$$

Show that $\Gamma \cup \Delta$ is consistent [Hint: use the compactness theorem].

(e) Show that if Γ is a set of sentences of any language and Γ has only infinite models, then Γ has infinite models of any cardinality [you may assume that a consistent set of sentences remains consistent if its language is extended by the addition of any cardinality of new constants].

8. A formula is said to be *prenex* when it is of the form

$$Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \alpha$$

where each Q_i is either $\forall$ or $\exists$ and α contains no quantifiers.

(a) Let L be a first order language with $\rightarrow$ as its only sentential connective. If α is a formula of L containing $n+1$ instances of the quantifier symbols, show (by induction on the degree of α) that there is a logically equivalent formula of the form $Qx\beta$ (where Q is $\forall$ or $\exists$) such that β contains only n occurrences of the quantifier symbols.
[You may assume the bound variables may be freely renamed and you may assume the following theorems: $\neg\forall x\phi \leftrightarrow \exists x\neg\phi$; $\neg\exists x\phi \leftrightarrow \forall x\neg\phi$. And provided provided x does not occur free in ψ, you may assume: $(\psi \rightarrow \forall x\phi) \leftrightarrow \forall x(\psi \rightarrow \phi)$; $(\psi \rightarrow \exists x\phi) \leftrightarrow \exists x(\psi \rightarrow \phi)$; $(\forall x\phi \rightarrow \psi) \leftrightarrow \exists x(\phi \rightarrow \psi)$; $(\exists x\phi \rightarrow \psi) \leftrightarrow \forall x(\phi \rightarrow \psi)$.]

(b) Conclude that for every formula of L there is a logically equlivalent prenex formula of L.

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9. An isomorphism h between two structures $\mathcal{A}$, $\mathcal{B}$ of a language L is a bijection $h : |\mathcal{A}| \mapsto |\mathcal{B}|$ (i.e. a 1-1 mapping from the universes of $\mathcal{A}$ and $\mathcal{B}$ to each other) such that for each n-ary predicate symbol P and each n-tuple $\langle a_1, \ldots, a_n \rangle$ of elements of $|\mathcal{A}|$,

$$\langle a_1, \ldots, a_n \rangle \in P^{\mathcal{A}} \text{ iff } \langle h(a_1), \ldots, h(a_n) \rangle \in P^{\mathcal{B}}$$

and for each n-place fynction symbol f and each such n-tuple,

$$h(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(h(a_1,) \ldots, h(a_n)).$$

(a) Show that for any quantifier-free formula α not containing the equality symbol,

$$\models_{\mathcal{A}} \alpha\ [s] \text{ iff } \models_{\mathcal{B}} \alpha\ [h \circ s]$$

[you may assume that for any term t: $h(\overline{s}(t)) = \overline{h \circ s}(t)$ where $\overline{s}(t)$ is computed in $\mathcal{A}$ and $\overline{h \circ s}(t)$ in $\mathcal{B}$]

(b) Let Γ be a set of sentences of a countable language such that Γ has only infinite models. Assume that any two models of Γ of cardinality κ are isomorphic. Use the upward and downward Löwenheim-Skolem theorems to conclude that Γ is a complete set of sentences.

10. Let $\mathbf{Cn}(\Gamma)$ be the deductive closure of Γ. A *Theory* is any set of sentences Σ such that $\Sigma = \mathbf{Cn}(\Sigma)$

(a) Argue that

i. $\Gamma \subseteq \mathbf{Cn}(\Gamma)$,

ii. $\Sigma \subseteq \Gamma \Rightarrow \mathbf{Cn}(\Sigma) \subseteq \mathbf{Cn}(\Gamma)$,

iii. $\mathbf{Cn}(\mathbf{Cn}(\Gamma)) = \mathbf{Cn}(\Gamma)$

(b) Define when a theory is

i. Axiomatisable

ii. Finitely Axiomatisable

iii. Decidable

iv. Complete

(c) Show that if $\mathbf{Cn}(\Sigma)$ is finitely axiomatisable, then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\mathbf{Cn}(\Sigma_0) = \mathbf{Cn}(\Sigma)$

(d) Argue that every complete, finitely axiomatisable theory is decidable.

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11. The theory N is presented by the finite axiomatization:

$(S.1)$ $\forall v_1(\mathbf{s}v_1 \neq 0)$

$(S.2)$ $\forall v_1 \forall v_2(\mathbf{s}v_1 = \mathbf{s}v_2 \rightarrow v_1 = v_2)$

$(L.1)$ $\forall v_1 \forall v_2(v_1 < \mathbf{s}v_2 \leftrightarrow v_1 \leq v_2)$

$(L.2)$ $\forall v_1(v_1 \nless 0)$

$(L.3)$ $\forall v_1 \forall v_2(v_1 < v_2 \vee v_1 = v_2 \vee v_2 < v_1)$

$(A.1)$ $\forall v_1(v_1 + \mathbf{s}_0 = v_1)$

$(A.2)$ $\forall v_1 \forall v_2(v_1 + \mathbf{s}v_2 = \mathbf{s}(v_1 + v_2))$

$(M.1)$ $\forall v_1(v_1 \times \mathbf{s}_0 = \mathbf{s}_0)$

$(M.2)$ $\forall v_1 \forall v_2(v_1 \times \mathbf{s}v_2 = (v_1 \times v_2) + v_1)$

Show that

(a) $N \vdash \forall x \neg(x < 0)$

(b) $N \vdash \forall v_1(v_1 < \mathbf{s}_{k+1} \leftrightarrow v_1 = \mathbf{s}_0 \vee v_1 = \mathbf{s}_1 \vee \ldots \vee v_1 = \mathbf{s}_k)$

(c) Show that for any closed atomic formula α in the language of N, either $N \vdash \alpha$ or $N \vdash \neg\alpha$. [You may assume that for any variable free term t there is a unique natural number n such that $N \vdash t = \mathbf{s}_n$.]

12. Suppose we have a function # assigning to each formula α of the language of arithmetic a (unique) integer $\#\alpha$, its Gödel number and a function $\mathcal{G}$ assigning to each finite sequence D of such formulas a unique integer $\mathcal{G}(D)$. Where N is as defined in question 11, show that there is a sentence σ such that neither $N \vdash \sigma$ nor $N \vdash \neg\sigma$.

[You may assume that #, $\mathcal{G}$ and the ternary relation R

$(a, b, c) \in R$ iff a is the Gödel number of some formula α and c is the value at $\mathcal{G}$ of some deduction of $\alpha(\mathbf{s}_b)$ from N

are representable in N.]

END OF PAPER