# UNIVERSITY OF LONDON

BA EXAMINATION

for Internal Students

This paper is also taken by Combined Studies Students

# PHILOSOPHY

Optional Subject (i): Set Theory and Further Logic

Answer THREE questions, at least ONE from EACH section

SECTION A

1. Prove or refute the following, in each case stating any axiom (schema) you use.

   (i) There is a set whose members are exactly the things that are not members of themselves: $\{x : x \notin x\}$.

   (ii) There is a universal set: $\{x : x = x\}$.

   (iii) There is an empty set: $\{x : x \neq x\}$. (You may assume that there is a set).

   (iv) For any set $A$ there is a set whose members are exactly the things that are not members of $A$: $\{x : x \notin A\}$.

2. (i) Define the Cartesian Product of $A$ with $B$, $A \times B$. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

TURN OVER

(ii) Define the following: $R$ is a relation; $R$ is an equivalence relation. Given that $R$ is an equivalence relation and that $x$ is in the field of $R$, define: the equivalence class of $x$ modulo $R$, symbolised $[x]_R$. Then prove that for any $z$ in the field of $R$, $z \in [x]_R$ iff $xRz$.

(iii) Let $R$ be an equivalence relation and let $A$ be the field of $R$. Prove that the set $\{[x]_R : x \in A\}$ partitions $A$, i.e. prove that

(a) for all $x$ in $A$, $[x]_R \subseteq A$ and $[x]_R \neq \emptyset$;
(b) for all $y$ in $A$, there is an $x$ in $A$ such that $y \in [x]_R$;
(c) for all $x$ and $y$ in $A$, $[x]_R \neq [y]_R$ implies $[x]_R \cap [y]_R = \emptyset$.

3. (i) Let $A$ be any non-empty set of sets. Let $R$ be the relation defined: $xRy$ if and only if $x \in A$ & $y \in A$ & there is a bijection from $x$ to $y$. Prove that R is an equivalence relation.

(ii) Let '$x \cong y$' abbreviate 'there is a bijection from $x$ to $y$'. Let $N$ be the set of non-negative integers. Let $E$ be the set of even members of $N$, and let $O$ be the set of odd members of $N$. Prove that (a) $E \cong O$, (b) $E \cong N$, and (c) $N \cong N - \{0\}$.

(iii) Prove Cantor's theorem: for any set $A, A \ncong \mathbf{P}A$, where $\mathbf{P}A$ denotes the power set of $A$.

4. Let $x^+$ be the von Neumann successor of $x$, namely $x \cup \{x\}$. Let

$Ind(A) \leftrightarrow_{df} \emptyset \in A \;\&\; \forall x[x \in A \rightarrow x^+ \in A]$,
$NN(x) \leftrightarrow_{df} \forall A[Ind(A) \rightarrow x \in A]$,
$\omega =_{df} \{x : NN(x)\}$.

Prove that

(i) $Ind(A) \rightarrow \omega \subseteq A$;

(ii) $Ind(A) \;\&\; A \subseteq \omega \rightarrow A = \omega$.

(iii) Let $\theta(x)$ be any 1-place predicate in the language of set theory. Assuming that $\{x : x \in \omega \;\&\; \theta(x)\}$ is a set, use (ii) to prove induction on $\omega$: If $\theta(\emptyset) \;\&\; \forall x \in \omega[\theta(x) \rightarrow \theta(x^+)]$, then $\forall x \in \omega\, \theta(x)$.

(iv) $Ind(\omega)$.

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5. (i) State the theorem that justifies definition by recursion on $\omega$. Let $k$ be any member of $\omega$. Define by recursion on $\omega$ a function $f_k$ whose value for any $n \in \omega$ is the result of adding $n$ to $k$.

(ii) Assuming that for each $k \in \omega$ there is a function $f_k$ as described in (i), define the binary addition function $x+y$ for $x, y$ in $\omega$. Then prove the equations

$x + \emptyset = x$;

$x + y^+ = (x + y)^+$.

(iii) Prove by induction on $\omega$ that $\forall n \in \omega, n = \emptyset$ or $\exists m \in \omega\ n = m^+$.

(iv) Prove that $\neg\exists n \in \omega, \emptyset = n^+$ .

6. (i) What is it for a binary relation $R$ to be (a) a partial ordering, (b) a total ordering, (c) a well-ordering of a set $A$? What is it for set to be a transitive set (not: transitive relation)? Show that every well-ordering is a total ordering, but not every total ordering is a well-ordering.

(ii) What is it for a set to be an ordinal? Given that any member of an ordinal is an ordinal and that any transitive set of ordinals is an ordinal, prove that there is no set of all ordinals.

(iii) Prove the least ordinal principle: For any 1-place predicate $\theta(x)$ in the language of set theory, if there is an ordinal $\alpha$ such that $\theta(\alpha)$, there is a least ordinal $\alpha$ such that $\theta(\alpha)$.

7. (i) Let $A \times B$ denote the Cartesian product of $A$ with $B$. Prove for any sets $K, L, M$, that (a) $K \times (L \cup M) = (K \cup L) \times (K \cup M)$, and (b) $K \times L \cong L \times K$

(ii) Define cardinal addition $\kappa + \lambda$. Prove

(a) $\kappa + \lambda = \lambda + \kappa$ and
(b) $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$.

(iii) Define cardinal multiplication $\kappa \cdot \lambda$. Prove

(a) $\kappa \cdot \lambda = \lambda \cdot \kappa$ and (b) $\kappa \cdot (\lambda + \mu) = (\kappa \cdot \lambda) + (\kappa \cdot \mu)$.

(iv) Prove (a) $\kappa + 0 = \kappa$ and (b) $\kappa \cdot 0 = 0$.

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8. (i) Assume that for any set $A$ there is an infinite cardinal greater than all in $A$. Define the aleph operator $\aleph$ by transfinite recursion on the ordinals, and show that for any ordinals $\alpha$ and $\beta$, $\alpha \in \beta \rightarrow \aleph_\alpha < \aleph_\beta$.

(ii) Prove that the predicate '$x = \aleph_y$ & $y$ is an ordinal' is univalent, i.e. that [$x = \aleph_{y_1}$ & $y_1$ is an ordinal, & $x = \aleph_{y_2}$ & $y_2$ is an ordinal] $\rightarrow y_1 = y_2$ .

(iii) State the Axiom schema of Replacement. Assuming that there is no set of all ordinals, use Replacement to show that there is no set of all alephs.

## SECTION B

9. Let three frames $(G_1, R_1), (G_2, R_2).(G_3, R_3)$, be defined by

$G_1 = \{a\},\ R_1 = \{\langle a, a\rangle\}$
$G_2 = \{a, b\},\ R_2 = \{\langle a, b\rangle, \langle b, a\rangle, \langle b, b\rangle\}$
$G_3 = \{a, b\},\ R_3 = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle\}$

($\langle x, y\rangle$ in $R_i$ means $xR_iy$). For each of the three fames determine which of the following formulas is/are valid on the frame.

(a) $\Box P \supset P$

(b) $\Box P \supset \Box\Box P$

(c) $P \supset \Box\Diamond P$

(d) $\Box P \supset \Diamond P$

Moreover, if one of the formulas $\phi$ is not valid on frame $(G_i, R_i)$, give a world $x$ in $G_i$ and a forcing relation $\vdash$ between $G_i$ and $\{P\}$ such that $x \vdash \neg\phi$.

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10. (i) Let '$\mathbf{C}P$' abbreviate 'It is contingently true that $P$'. Define $\mathbf{C}P$ in terms of one of the standard modal operators and explain why $\mathbf{C}P \supset \Box \mathbf{C}P$ is unacceptable.

    (ii) What are the modal systems S4 and S5? Under what conditions on the accessibility relation on a model $M$ for propositional modal logic is $M$ a model of (a) S4, (b) S5?
    Show that $\Diamond\Box P \supset \Box P$ is true at every possible world of any S5 model but false at some possible world of some S4 model.

11. Use the propositional tableau proof systems to prove the following three formulas

    (a) $\neg(\Box\phi \wedge \Diamond\neg\phi)$ in the $\mathbf{T}$ system
    (b) $\Diamond(\Diamond\phi \vee \Diamond\psi) \supset (\Diamond\phi \vee \Diamond\psi)$ in the system $\mathbf{S}4$,
    (c) $\neg(\Diamond\neg\phi \wedge \Diamond(\Box\phi \wedge \psi))$ in the system $\mathbf{S}5$,

    Use the constant domain tableau system to determine whether (d) is valid on all $\mathbf{K}$ models with constant domain

    (d) $(\exists x)\Diamond((\exists x)\Diamond P(x) \supset \Diamond P(x))$

    Use the variable domain tableau system to determine whether (e) is valid on all $\mathbf{K}$ models with varying domain

    (e) $(\exists x)\Diamond(\neg P(x) \vee (\forall x)\Box P(x))$.

12. Let $(G, R)$ be the following frame: $G = \{0, 1, 2, 3 \ldots\}$, the set of natural numbers, and $nRm$ holds if $n$ is a smaller number than $m$. Let $(G, R, \Vdash)$ be a model over $(G, R)$. In such a model, '$n \Vdash P$', means that natural number $n$ has some property $\phi_P$, that is, the model interprets $P$ as a specific property of natural numbers (eg., "$n$ is smaller than a given number $a$", "$n$ is a multiple of 3", "$n$ is $4, 7$ or 18", etc.).

    (i) For each of the following formulas give a property $\Phi_P$ of natural numbers such that the formula is valid on the model $(G, R, \Vdash)$
       (a) $P \supset \Box\Diamond P$
       (b) $P \supset \Box P$
       (c) $\Box(P \supset \Box\neg P)$

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(ii) Argue that $(\Diamond P \wedge \Diamond Q) \supset \Diamond((\Diamond P \wedge Q) \vee (P \wedge \Diamond Q) \vee (P \wedge Q))$ is valid on the frame $(G, R)$

(iii) Let $nQm$ mean that number $n$ is larger than number $m$. Give a formula that is valid on the frame $(G, Q)$ but not on $(G, R)$.

13. In a model $(G, R, \vdash)$ over the frame $(G, R)$ from the previous question, let the function $D$ assign to every element of $G$ a *non-empty* domain and let the function $v$ assign to every variable an element of the domain of the model.

(i) Argue that, if $D(0)$ is a *finite* domain and all Barcan formulas are valid on this model, then $\Diamond\Box\phi$ is valid, for $\phi$ any converse Barcan formula.

(ii) Suppose $(G, R) \vdash_v \phi(y)$, where $\phi(y)$ is one of the formulas

(a) $\Box\Diamond(\exists x)(x = y)$,

(b) $\Diamond\Box(\exists x)(x = y)$,

(c) $(\exists x)(x = y) \supset (\exists x)\Box(x = y)$.

For each of the formulas, explain what properties the domain element $v(y)$ satisfies.

(iii) Let $D$ assign the same non-empty domain to every natural number and let $\phi(y)$ be a property of natural numbers, such that $(\forall y)(\Box\Diamond\phi(y) \supset (\phi(y) \wedge \Box\phi(y))$ is valid on the model. What does this say about the property $\phi(y)$? (hint: as any implication $(P \supset Q)$ is logically equivalent to $(\neg P \vee Q)$, there are two cases to check).

END OF PAPER