

Set Theory questions for the BA paper in SET THEORY AND FURTHER LOGIC 2007

1. (i) Present Russell's Paradox. How is the argument avoided in Axiomatic Set Theory?
 - (ii) Prove that there is no universal set.
 - (iii) Prove that there is no set of all unit sets.
 - (iv) Assuming that there is a set, prove that there is exactly one empty set.
 - (v) Prove that every set has a subset.

2. (i) What is (a) a function, (b) an injection, (c) a bijection from A to B , (d) the identity function on A , id_A (e) f^{-1} , where f is an injection, (f) $g \circ f$, where $\text{ran}(f) \subseteq \text{dom}(g)$?

Let f be an injection with domain A and range B . Show that

- (ii) f^{-1} is a bijection from B to A .
- (iii) $f^{-1} \circ f = \text{id}_A$.
- (iv) Let $x \preccurlyeq y$ if and only if there is an injection with domain x and range a subset of y . Show that the relation \preccurlyeq on the class of sets is reflexive, transitive but not symmetric.

3. Let $x \approx y$ if and only if there is a bijection from x to y . Show that the relation \approx is reflexive, symmetric and transitive. Let \preccurlyeq be defined as in (iv) of the previous question. Prove the Schröder-Bernstein theorem: for any sets x and y , if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x \approx y$.

4. (i) Define the operations of cardinal addition $\kappa + \mu$ and cardinal multiplication $\kappa \cdot \mu$, and show that $\kappa \cdot \mu = \mu \cdot \kappa$ for cardinals κ, μ .

Let κ, λ, μ be any cardinals. Show the following:

- (ii) $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$
- (iii) If $\lambda \leq \mu$, $\kappa + \lambda \leq \kappa + \mu$.
- (iv) $(\kappa + \lambda) \cdot \mu = (\kappa \cdot \mu) + (\lambda \cdot \mu)$.

5. (i) What is it for a binary relation R to be a (a) *partial ordering*, (b) a *total ordering*, (c) a *well-ordering* of a set A ?
- (ii) Give an example of a set with a total ordering on it which is not a well-ordering.
- (iii) Which of the kinds of ordering mentioned in (i) is exemplified by the standard ordering $<_{\mathbb{N}}$ of the set \mathbb{N} of natural numbers?
- (iv) Let R be a binary relation on a set A and let S be a binary relation on a set B . Under what conditions is the ordered set $\langle A, R \rangle$ *isomorphic* to the ordered set $\langle B, S \rangle$?
- (v) Let \mathbb{E} be the set of even natural numbers and $<_{\mathbb{E}}$ their natural ordering. Show that $\langle \mathbb{E}, <_{\mathbb{E}} \rangle$ is isomorphic to $\langle \mathbb{N}, <_{\mathbb{N}} \rangle$.
6. (i) Show that for any set A , $|A| < |PA|$, where for any set X , $|X|$ is its cardinal number and PX is the power set of X .
- (ii) If $B \subseteq A$, what is the *characteristic function of B in A* ? Define cardinal exponentiation κ^{μ} for cardinals κ, μ . Show that for any set A , $|PA| = 2^{|A|}$.
- (iii) Let \mathbb{I} be the set of infinite decimals between 0 and 1. Let \mathbb{N} be the set of natural numbers. Show that $|\mathbb{N}| < |\mathbb{I}|$.
- (iv) Show that if κ is an infinite cardinal, $\kappa + \kappa = \kappa$. Say (without proof) why it is not the case that $\kappa^{\kappa} = \kappa$?
7. State the Axiom of Choice. Assuming that the class of ordinals is not a set, use the Axiom of Choice and definition by transfinite recursion on the ordinals to prove that every set is equipollent to an ordinal. Hence show that for any set A there is a well-ordering on A .
8. (i) Define the aleph function \aleph_{α} . Show that for any ordinal α , $\aleph_{\alpha} \notin \{\aleph_{\gamma} : \gamma < \alpha\}$.
- (ii) Prove that for any ordinals α and β , if $\alpha < \beta$ then $\aleph_{\alpha} < \aleph_{\beta}$.
- (iii) Prove that for every ordinal α , \aleph_{α} is an infinite cardinal.
- (iv) Assuming that there is no set of all alephs, prove that every infinite cardinal is an aleph, i.e. that for every infinite cardinal number λ , there is an ordinal α such that $\lambda = \aleph_{\alpha}$.