UNIVERSITY OF LONDON

MA EXAMINATION 2005

for Internal Students

SYMBOLIC LOGIC

Candidates should answer THREE of the following questions.
Please avoid overlap in your answers.

1. i. For Σ a set of sentences of Sentential Logic it holds that "if $\Sigma \models \phi$, then there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \phi$". Use this to prove the compactness theorem.

   ii. Assume that every finite subset of Σ is satisfiable. Show that, for any propositional sentence α, the same holds for at least one of $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$.

   iii. Let Δ be a set of sentences of Sentential Logic such that every finite subset of Δ is satisfiable and for every sentence ϕ, $\phi \in \Delta$ or $\neg\phi \in \Delta$. Define the truth valuation v by $v(P) = \top$ if $P \in \Delta$ and $v(P) = \bot$ if $P \notin \Delta$ for each propositional variable P. Show that for every sentence ϕ we have: $\overline{v}(\phi) = \top$ iff $\phi \in \Delta$.

2. Suppose $\phi_1, \ldots, \phi_n$ is a deduction of $B \rightarrow Ax$ from Φ where A is a unary predicate symbol and the variable x does not occur free in Φ or B. Prove that $\Phi \vdash B \rightarrow \forall x Ax$ by induction on n (i.e., show that $\Phi \vdash B \Rightarrow \forall x \phi_k$ for all $k, 1 \leq k \leq n$.).
   In your proof you may use all tautologies plus the axioms

   (i) $\forall x(\alpha \rightarrow \beta) \rightarrow (\forall x\alpha \rightarrow \forall x\beta)$

   (ii) $\alpha \rightarrow \forall x\alpha$, for x not free in α

   (iii) $\forall x_1 \forall x_2 \ldots \forall x_n \alpha$ for any variables $x_1 \ldots x_n$ and α any axiom of the Predicate Calculus.

TURN OVER

3. Let Γ be a set of sentences in the language of arithmetic that is maximal w.r.t. the property that $\Gamma \nvdash \alpha$ for some fixed α (maximality means here that if $\beta \notin \Gamma$ then $\Gamma \cup \{\beta\} \vdash \alpha$).

   (i) Show that Γ is complete.

   (ii) Given that $A_E \subseteq \Gamma \subseteq Th(\mathfrak{N})$, outline a proof that Γ is not axiomatizable (where $Th(\mathfrak{N})$ is the set of sentences true on the standard structure of natural number and A_E is a finitely axiomatized theory in the first-order language of arithmetic, such that every recursive relation is representable in A_E).

4. Outline a proof of Tarski's Undefinability Theorem and discuss its connection with Gödels Incompleteness Theorem.

5. Given that A_E is a finitely axiomatized theory in the first-order language of arithmetic, such that every recursive relation is representable in A_E, prove:

   i. For any recursive relation R there is a formula that represents R in any theory Σ in the language of arithmetic such that $A_E \cup \Sigma$ is consistent.

   ii. If Σ is a theory in the language of arithmetic such that $A_E \cup \Sigma$ is consistent, then Σ is not recursively decidable.

6. i. Let Γ be a recursive set of sentences. Argue that the relation $\text{Ded}_\Gamma(x, y)$ given by

   $Ded_\Gamma(x, y) \iff x$ is a the code of a sentence and y is the code of sequence-of-formulas that constitutes a deduction of that sentence from Γ,

   is recursive.

   ii. Outline a proof that a relation P is recursively enumerable if and only if it is weakly representable in A_E.

TURN OVER

7. (i) Show that there are no sets y and z such $\forall x[x \in y \leftrightarrow x \notin z]$.

(ii) Show that for any sets x and y there is a set whose members are exactly the ordered pairs whose first member is in x and whose second is in y.

(iii) Show that for any sets x and y there is a set of all functions from x to y.

(iv) Define $xRy \leftrightarrow [x = y \ \& \ \theta(y)]$, where $\theta(y)$ is any formula in the language of set theory with only y occurring free. Show that, for any set s there is a set t such that $\forall y[y \in t \leftrightarrow \exists x(x \in s \ \& \ xRy)]$. Use this to show that the Axiom Schema of Separation is deducible from the Axiom Schema of Replacement.

8. (i) Show that there is a set N, a function s from N to N, and a member e of N such that the following three conditions hold: (a) $e \notin ran(s)$, (b) s is injective, (c) for any subset A of N, if $e \in A$ and $s(x) \in A$ whenever $x \in A$, then $A = N$.

(ii) State the theorem that legitimates definition by recursion on ω. Use it to define binary functions on ω for addition and multiplication in ω.

(iii) Give a set theoretic definition of the set I of binary expansions of real numbers r such that $0 \leq r \leq 1$. Show that $|I| = |P\omega|$, i.e. that the cardinality of I is the cardinality of the power set of ω.

(iv) Show that $|P\omega| = 2^{|\omega|}$ and that $2^{|\omega|} > |\omega|$.

9. (i) State the theorem that legitimates definition by (transfinite) recursion. Prove that the axiom of choice entails the well-ordering theorem: for every set A there is a well-ordering on A.

(ii) Deduce from the well-ordering theorem cardinal comparability: for any sets A, B, $|A| \leq |B|$ or $|B| \leq |A|$.

(iii) Show that the well-ordering theorem entails the axiom of choice.

TURN OVER

10. (i) Specify the systems of sentential modal logic (SML) known as T, Br, S4, S5. For any two systems of SML X and Y, let '$X < Y$' mean that every theorem of X is a theorem of Y but not every theorem of Y is a theorem of X. Show the following:

    T < S4 < S5. (b) T < Br < S5.

    (ii) Give Kripke semantics for the language of SML. (a) Show that there is a condition C_1 on the accessbility relation R of a Kripke interpretation such that every T-theorem, but not every S4-theorem, is true under any Kripke interpretation in which R satisfies C_1. (b) Show that there is a condition C_2 on the accessibility relation R such that every S4-theorem, but not every S5-theorem, is true under any Kripke interpretation in which R satisfies C_2. (c) Show that there is a condition C_3 on the accessibility relation R such that every S5-theorem is true under any under any Kripke interpretation in which R satisfies C_3.

    (iii) (a) Show that there is a condition C_4 distinct from the conditions C_i that you have so far specified such that $(P \rightarrow \Box\Diamond P)$ is true under any Kripke interpretation in which the accessibility relation R satisfies C_4. (b) Is the formula $(P \rightarrow \Box\Diamond P)$ a theorem of S4? Verify your answer.

11. What is a 'possible worlds' model for quantified modal logic that allows domains to vary over worlds? With respect to such models specify the semantic rules (i.e. clauses in the truth-definition) for atomic sentences, quantified sentences, and modal sentences. Let @ be the actual world. Let '$\mathbf{A}P$' (for 'Actually P') be interpreted according to the rule: $\mathbf{A}P$ is true at w in model M if and only if P is true at @ in M.

    (i) Let a one-place predicate $E^*(x)$ be defined thus: $E^*(x) \leftrightarrow \exists y(y = x)$. Describe a model in which $\exists x \Diamond \neg E^*(x)$ is true at @. Justify your answer.

    (ii) Show that there is a model M for quantified modal logic with the same domain at every world, such that for some world w other than @ the value of $\iota x P x$ at w in M differs from the value of $\iota x \mathbf{A} P x$ at w in M.

TURN OVER

(iii) Consider models for quantified modal logic with the same domain at every world, such that $\exists x(Px \;\&\; \forall y(Py \supset y = x)$ is true at every world. Answer with justification the following: (a) Is $\Box(\iota x Px = \iota x Px)$ true at @ in every such model? (b) Is $\Box(\iota x Px = \iota x \mathbf{A} Px)$ true at @ in every such model? (c) Under what circumstance is $\iota x Px$ a rigid designator in such a model? If $\iota x Px$ is a rigid designator in such a model, is $\Box(\iota x Px = \iota x \mathbf{A} Px)$ true at @ in that model?

(iv) Formalise the following in the language of quantified modal logic with the 'actually' operator: (a) There could have been a giraffe taller than the actual tallest giraffe. (b) The Eiffel Tower could have been taller than it actually is.

(v) Use the resources of quantified modal logic to respond to the following:

> Mathematicians may conceivably be said to be necessarily rational and not necessarily two-legged; and cyclists necessarily two-legged and not necessarily rational. But what of an individual who counts among his eccentricities both mathematics and cycling? Is this concrete individual necessarily rational and contingently two-legged, or vice versa? Just insofar as we are talking referentially of the object ... there is no semblence of sense in rating some of his attributes as necessary and others as contingent.
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> From Quine, *Word and Object.*

END OF PAPER